Wallace's Theorem Concerning Plane Polygons of the Same Area.

By W. H. Jackson.

The object of this paper is to draw attention to Wallace's theorem that *two plane polygons of the same area can in all cases be divided into respectively congruent portions; they are congruent by dissection.* Every student of elementary geometry ought to know this fact, not only on account of its intrinsic interest but also on account of the stimulus it affords towards getting an experimental knowledge of the subject.

This theorem was stated by William Wallace about the year 1807 in the new series of Leybourne's *Mathematical Repository*, Vol. III. Later, in 1831, it appeared in the form of an addition to Playfair's edition of Euclid's Elements.* It is there proved that each construction in Euclid's chain yields figures which are congruent by dissection as well as of the same area. No mention of Theorem I of the present paper is made, though it is necessary for a complete proof.

The present writer was led to the final result by exactly the same steps as those exhibited in Wallace's note. But Wallace adds a corollary that two plane polygons of equal area can in all cases be divided into the *same number of* respectively congruent *triangles.* The rest of this paper remains as it was before a hint given by Dr. F. Morley led to the discovery of Wallace's work. Only the proofs of Theorems IV and VI appear to be novel, but conciseness is gained by seeking as direct a proof of Wallace's theorem as may be possible.

* "Elements of Geometry"; containing the first six books of Euclid; by John Playfair, with additions by William Wallace: eighth edition, Edinburgh, 1831.

The fundamental theorems concerning areas may be divided into three groups:

(1) The theorems leading to the determination of a square of the same area as any given polygon.

(2) The theorem of Pythagoras.

(3) The theorems corresponding to the distributive law in algebra.

In the usual American and English text-books the proofs of these theorems are probably intended to be based, so far as area is concerned, on the following axioms:

(1) The area of a geometrical figure is a number.

(2) Congruent figures have the same area.

(3) The area of a figure is the sum of the areas of its parts.

(4) If equal areas be added to equal areas, the wholes are of equal area.

(5) If equal areas be taken from equal areas, the remainders are of equal area.

(6) The halves of equal areas are of equal area.

The first three axioms are not always specifically stated. They are of themselves quite sufficient for the basis of the theory of area, for the last three readily follow.

If we turn to Weber and Wellstein's Encyclopædia* of elementary mathematics, we find a quite different treatment. In the first place the algebraic theory of number is dispensed with by means of the purely geometrical method of comparing ratios of straight-line segments. In the second place no use of the axiom of Archimedes is made, so far as area is concerned.

With regard to the first point, if in the propositions briefly indicated above we everywhere replace the phrases "equal in area" or "of the same area" by "congruent by dissection," there is then no occasion for the introduction of a numerical measure of area.

With regard to the second point, Hilbert has shown † that it is impossible to prove that two triangles on the same base and of the same height are necessarily congruent by dissection without assuming the axiom of Archimedes.

---

* *Encyklopädie der Elementar-Mathematik*, Vol. II, p. 254, 2nd edition, Leipzig, 1907.

† "Grundlagen der Geometrie."

Any theory of area which omits this axiom is therefore necessarily incomplete. Though Weber and Wellstein prove, assuming the axiom of Archimedes, that triangles on the same base and of the same height are congruent by dissection, they do not state that it follows from this that all polygons of the same area are necessarily congruent by dissection. Incidentally, it may be noted that the rôle played by the axiom of Archimedes in the works last quoted, where it follows the axioms of congruence, is nothing more or less than to exclude null segments.

DEFINITION. *Two figures are congruent by dissection when either can be divided into parts which are respectively congruent with the corresponding parts of the other.*

THEOREM I. *Two figures each congruent by dissection with a third are congruent by dissection with each other.*

Suppose that figures A and B are each congruent by dissection with C. Accordingly let C be composed of portions A'_r $(r = 1, 2, \ldots, m)$ each congruent with the corresponding member of the series of parts A_r $(r = 1, 2, \ldots, m)$ which compose A. Similarly let C be composed of portions B'_r $(r = 1, 2, \ldots, n)$ congruent with the corresponding portions B_r $(r = 1, 2, \ldots, n)$ which compose B.

Let $[A'_r B'_s]$ denote the portion common both to A'_r and B'_s. Since A_r is congruent with A'_r it can be dissected into portions respectively congruent with $\sum_{s=1}^{n} [A'_r B'_s]$, and A is therefore composed of portions respectively congruent with the parts of C denoted by $\sum_{r=1}^{m} \sum_{s=1}^{n} [A'_r B'_s]$. But B can in the same way be divided into portions congruent respectively with these same parts.

That is, A and B are congruent by dissection.

This mode of proof is clearly not confined to polygons. It is not even confined to surfaces, but applies equally to volume spaces.

THEOREM II. *A parallelogram can be found congruent by dissection with any given triangle.*

Choose as base the longest side of the triangle and make one adjacent side of the parallelogram just half an adjacent side of the triangle, so that the triangle and the parallelogram have an *acute* angle in common. [The details are arranged so that by means of Theorems II and III a triangle can in *every* case be divided into three parts congruent with corresponding parts of a rectangle.]

THEOREM III. *A rectangle can be determined congruent by dissection with any given parallelogram.*

Choose as base a side of the parallelogram which is not less than either adjacent side.

THEOREM IV. *A rectangle can be found on any given base congruent by dissection with any given rectangle.* (Fig. 1.)

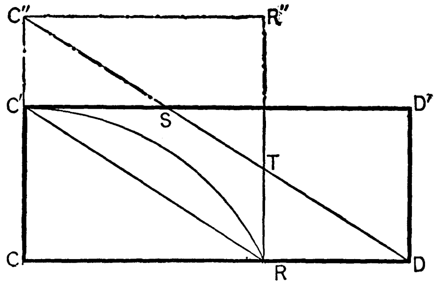

FIG. 1.

Let $ABB'A'$ be the given rectangle and let PQ be the given base. One of three cases must arise:

$$\text{(i)} \quad AB > 2PQ,$$
$$\text{(ii)} \quad 2PQ \overline{\gtrless} AB \geqq PQ,$$
$$\text{(iii)} \quad PQ > AB.$$

In case (i) let $CDD'C'$ be the first rectangle congruent by dissection with $ABB'A'$ and such that $2PQ \geqq CD$, obtained by successively halving the base and consequently doubling the height. By hypothesis $2CD$ does not satisfy the first inequality; that is, $CD > PQ$, and conditions (ii) are satisfied by the new base CD.

Similarly, by successively doubling the base and consequently halving the height we can also reduce the consideration of case (iii) to that of case (ii).

In case (ii) the theorem can be proved as follows: If on CD, $C'D'$ we choose lengths CR, SD' each equal to PQ, the straight line through R parallel to CC' necessarily meets SD in some point T within the rectangle when conditions (ii) are satisfied. Let DS, CC' when produced meet in C''. Then if $CRR''C''$ be a rectangle, it is the rectangle required, congruent by dissection with the given rectangle $ABB'A'$.

First, the rectangle $CRR''C''$ exists in all cases.

Secondly, $CRR''C''$ and $CDD'C'$ are congruent by dissection because the part $CRTSC'$ is common to both, $C'SC''$ and RDT are congruent, $C''TR''$ and SDD' are congruent.

Lastly, by Theorem I $CRR''C''$ is also congruent by dissection with $ABB'A'$.

THEOREM V. WALLACE'S THEOREM. *Two plane polygons of the same area are congruent by dissection.*

First, any polygon can be divided into triangles, and by Theorems I–IV rectangles can be found on a given base respectively congruent by dissection with all these triangles; these rectangles can be so placed as to form a single rectangle on the given base congruent by dissection with the given polygon.

Secondly, if two polygons of the same area are given, there are two rectangles on any given base respectively congruent by dissection with the two polygons, and therefore the two rectangles also have the same area by axiom (3). They must also be congruent; for, if not, one would have an area less than that of the other. It now follows from Theorem I that the two given polygons are congruent by dissection.

COR. I. If two polygons congruent by dissection are taken from polygons congruent by dissection, the remainders are congruent by dissection.

COR. II. The halves of polygons congruent by dissection are congruent by dissection.

THEOREM VI. *A square can be found congruent by dissection with any given rectangle.* (Fig. 2.)

Let $ABB'A'$ be the given rectangle, where $AB > AA'$. Two cases arise according as:

$$\text{(i)} \quad AB > 4\,AA',$$

$$\text{(ii)} \quad 4\,AA' \gtreqless AB > AA'.$$

As in the last theorem, the consideration of case (i) can easily be led back to case (ii) by halving the base of the rectangle and doubling its height, and repeating the process as often as may be necessary.

Let $CDD'C'$ be a rectangle congruent by dissection with $ABB'A'$ satisfying conditions (ii). From $C'C$ produced suppose CE taken congruent with CD and let O be the center of $C'E$. The circle with center O and radius OC' meets CD in some point R, lying in CD and not in CD produced, because $CR < CO + OR$, and therefore $CR < CD$. With the same construction as in Theorem IV, let SD' in $C'D'$ be congruent with CR and let DS, CC' produced meet in C'' and let $CRR''C''$ be a rectangle.

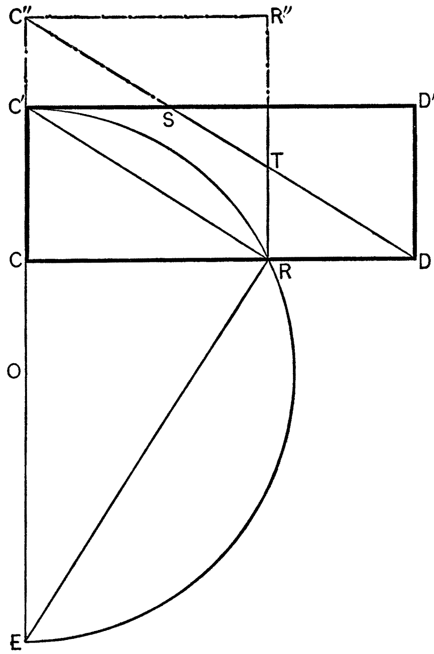

FIG. 2.

It is easily proved (if desired, without depending on the similarity of triangles) that, by virtue of conditions (ii), $2\,CR \geqq CD$. Therefore RR'' meets SD in some point T, and the rectangle $CRR''C''$ is congruent by dissection with the given rectangle exactly as in the last theorem.

Finally, since the angles CDC'', REC' are congruent, it follows that CC'', CR are congruent. The desired square has therefore been found.

THEOREM VII. *A square can always be found congruent by dissection with any given polygon.*

As in Theorem V a rectangle can be found on any given base congruent by dissection with a given polygon. By Theorem VI a square can be found congruent by dissection with this rectangle, and by Theorem I this square is congruent by dissection with the given polygon.

In conclusion, two questions naturally arise:

(i) *How many subdivisions are in general necessary to exhibit the congruence of two polygons of the same area containing given numbers of sides?*

That it is impossible to fix an upper limit to the number of subdivisions necessary to exhibit the congruence by dissection of two polygons, merely being given the number of sides each contains, will be clear from a simple example.

A rectangle 100 by .01 is congruent by dissection with a unit square. The straightforward way to exhibit this congruence is to divide the longer side of the rectangle into 10 equal parts, placing the 10 rectangles so obtained together so as to form a rectangle 10 by .1, and then to repeat the process in order to obtain the unit square. This method involves 18 cuts dividing the rectangle into 100 parts. If the construction of Theorem VI be followed, the longer side of the rectangle would be halved 6 times and the construction would be completed by 2 more cuts. This method involves 8 cuts dividing the rectangle into 164 parts. In either method the number of parts depends on the shape of the rectangle.

The maximum number of cuts necessary to transform a triangle into a rectangle is 2, dividing it into 3 parts; at least 2 more cuts are necessary to obtain a rectangle on any given base, and the least number of resulting parts is 5.

(ii) *Why should the results obtained with regard to polygons not be capable of extension to curved plane figures or to polyhedra?*

A necessary condition that a polygon should be congruent by dissection with a rectangle is that the sum of its angles must be an integral multiple of a straight angle. This condition is, of course, always satisfied, but the analogous condition is not satisfied for curved plane figures or for polyhedra.

The analogous condition that a curved figure may be congruent by dissection with a rectangle is that the sum of all its angles is an integral multiple of a straight angle and that all its curved arcs should fall into pairs of congruent portions only differing as to the side on which the interior of the figure lies.

The corresponding necessary, but not necessarily sufficient, condition that a polyhedron may be congruent by dissection with a rectangular block is that such of its dihedral angles as bear incommensurable ratios to a straight angle must fall into groups such that their sum, when multiplied by integers respectively inversely proportional to the lengths of their edges, bears a commensurable ratio to a straight angle.

For example, *a tetrahedron containing three intersecting edges mutually perpendicular and of lengths* 1, $\sqrt{2}$, $\sqrt{2}$ *is congruent by dissection with a cube.* Its dihedral angles all bear commensurable ratios to a straight angle. Similarly, *two polyhedra symmetric with respect to a point or to a plane are congruent by dissection.*

MANCHESTER, ENGLAND, *September*, 1911.